The research of irregular parquet forming convex pentagons for constructing monoendric geometrical networks is a special case in the infinite area of networks.

Unlike regular hexagons, regular pentagons do not form monoendric networks, yet irregular pentagons neatly compensate for this shortage by delivering 15 parquet pattern types (Figure 1).

As any triangle and tetragon are parquet forming (diagonally or translationally), and the last parquet convex polygon, in many forms of it, is the hexagon, so our attention is naturally turned towards pentagons and hexagons.

As a plastic phenomenon, the pentagon, due to its smaller number of sides and angles, bears a more typical and categorical form as compared to the more "worn out" hexagon. The problem of "wearing out" becomes clearer if we'd look at irregular polygons with larger number of sides.

This is even obvious with regular polygons, which, at number of the sides n = ∞, to wear out a circle. This explains our preference for pentagons in parquet forming modules. There is no question about their parqueting possibilities.

The problem of constructing the networks that reduce to surface ornament is in fact as old as art itself. The palace of Alhambra (XII – XIV century), which inspired Escher, is one of the peaks of a long development. Much later in the XVII century, the first attempt at mathematical analysis of geometrical networks by Kepler (1571–1630) – Harmonice Mondi (1619) emerged, but this work was overshadowed by his astronomical discoveries.

And so it was, till the end of the XIX century.

Even newer for the science is the problem of parquet forming pentagons. It emerged in 1918 when Reinhardt discovered the three types of parquet forming hexagons and the five types of pentagons. Quite later in the ‚Mathematical Games' column of the Scientific American Magazine, Martin Gardner attracted the attention of Kerschner, who discovered three more types, and of James, who gifted the world with his magnificent network (IX). Gardner's circle was also joined by Doris Schattschneider who summarized the experience of her predecessors, and by Marjory Rice.

Based on an analysis of the already existing parquet forming pentagons, Rice created a system of her own for visual information, revealing the different types of knots in a network, as follows:

I – link of two angles = 180º;

Λ – link of 3 angles = 360º;

Y – link of 3 angles (one is repeated) = 360º;

X – link of 4 angles (one is repeated) = 360º;

L – right angle repeated = 180º.

Yet the most important here is her formula: **All angles of an infinite network generating pentagon must present at all its knots in an equal number**. Based on what this number was, she divided the possible networks into degrees – 1^{st}, 2^{nd},…, N^{th}. With combinations of the different knots following the above formula tables were composed of possible parquet forming pentagons (with the relevant modelling of their sides).

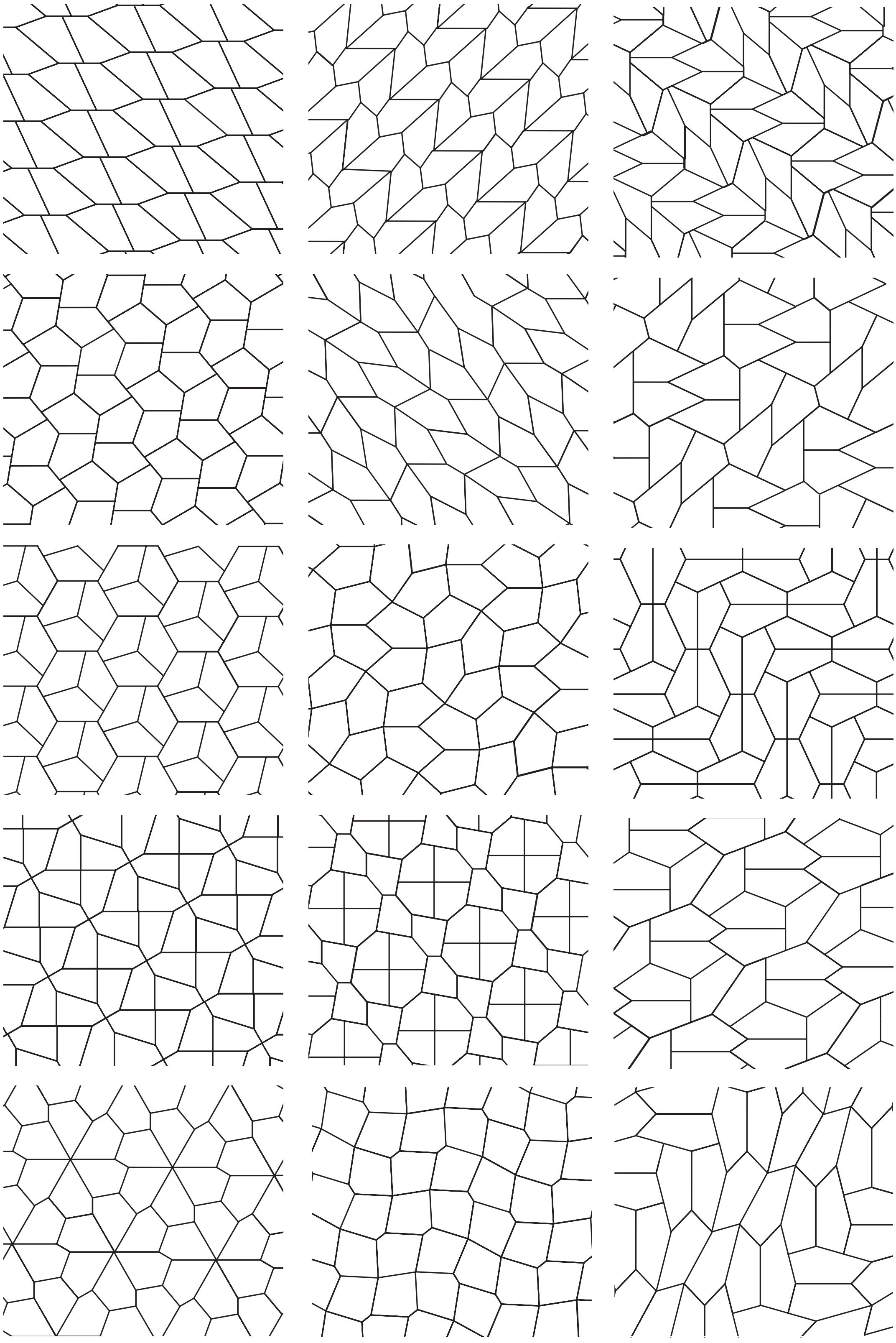


Figure 1: The 15 revealed so far parquet forming types of convex irregular pentagons

So the search for parquet pentagons was no longer accidental or the result of some ingenious idea (James), but a patient research after a determined program. Thus, in the first degree (where each angle is presented once), fall networks I and II. Network I poses no requirements for the sides, but Network II does (Figure 2).

All other networks, excluding III, V and IX, fall in the table of second degree, and there are 10 possible groups, as follows:

ΛΛII   ΛΛLL   ΛΛIL   ΛYII   YYII
YYIL   YYLL   ΛΛX   ΛYX   YYX

Each of these groups divides into different numbers of subgroups. For example, group YYII gives twelve subgroups, and group ΛΛLL – only two. (Figure 3).

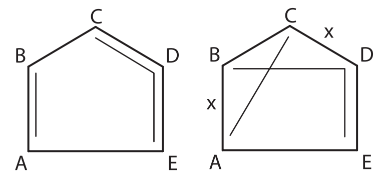


Figure 2

Marjory Rice composed tables, including all subgroups of these groups, forming the second degree, and consecutively checked each pattern from the subgroup to see if and how it forms parquet. By methodically excavating the multiple layers of possibility, new patterns were revealed, parquet X of the YYX group. A bit later, using the same methodical approach, plus the impulses received after she studied the theory of Branko Gruenbaum and Jeffrey Shepherd on block parquet forming, Marjory Rice discovered also networks XI and XII – both in one subgroup of the YYIL group, and finally – network XIII from the YYLL group.

With this, it seemed that actually all the thirteen types of parquet forming pentagons have been discovered. At least so stated mathematicians Hirshhorn and Hant (Mathematical Magazine, 1978). Here, it should be pointed out though, that network VI is a bimorph one. With this, the problem with the number of parquet forming types seemed to be settled, yet it was once again opened in 1985 after Roll Stein discovered type XIV…And again in 2015 professor Mann from the Washington University discovers type XV./It should be noted that the above type as well as type XIV have fixed angles and sides./ So the problem of searching for new parqueting type still remains open. Those types should be searched for within the framework given by Rice and her formula.

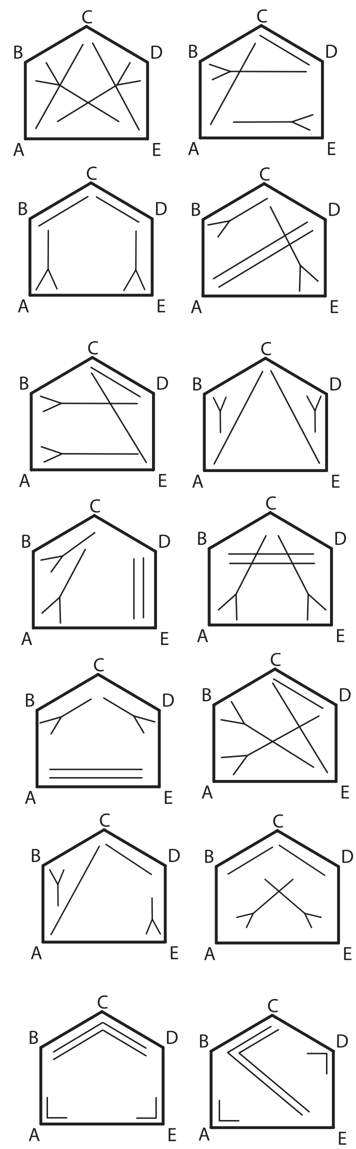
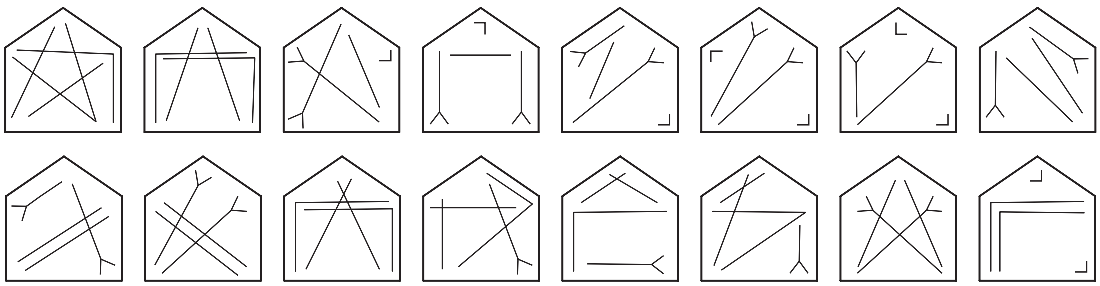


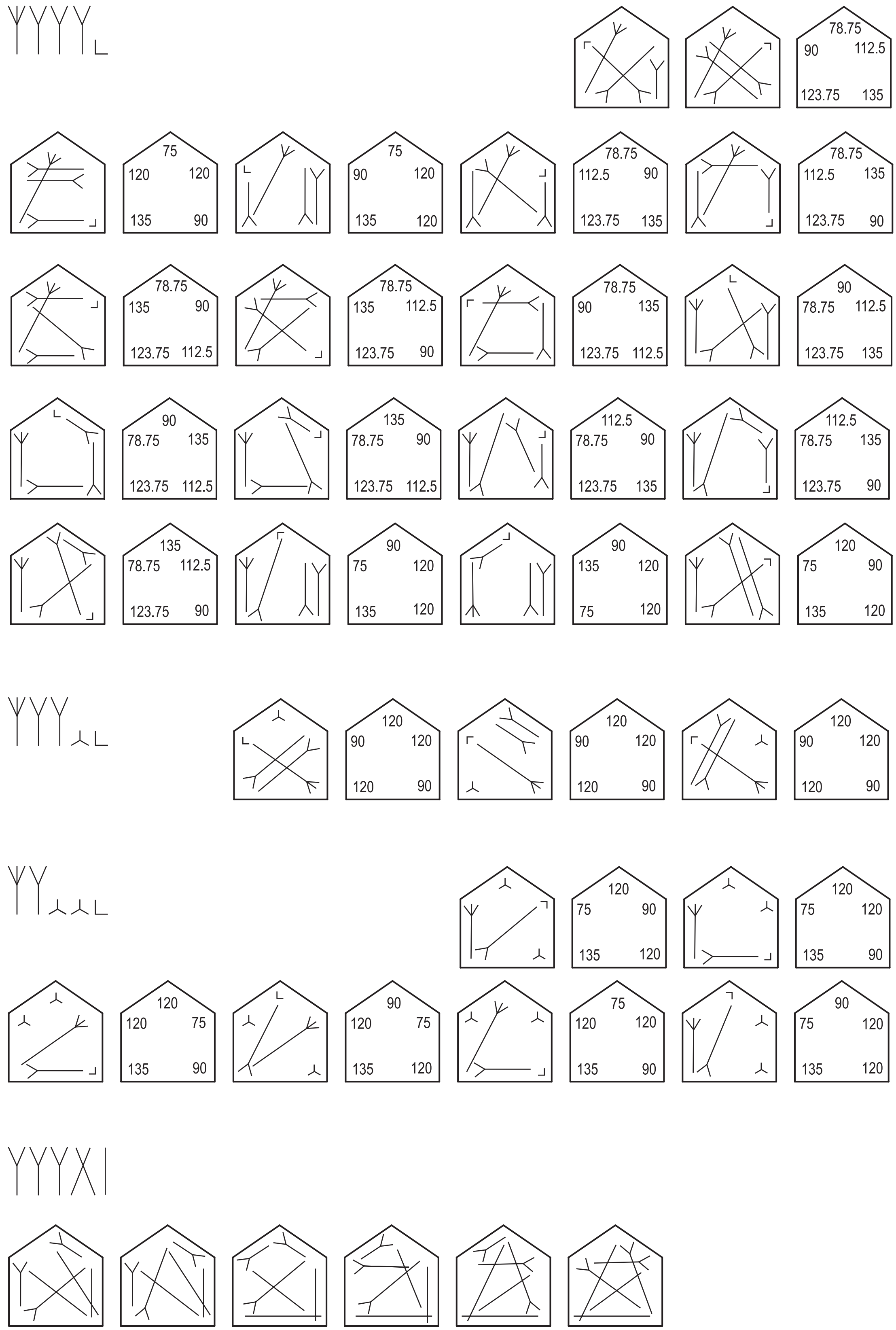
Figure 3

TABLE 1. THE POSSIBILITY FOR NEW PARQUETING TYPES

II DEGREE



III DEGREE



78.75 90 112.5 123.75 135
75 120 120 135 90
75 90 120 135 120
78.75 112.5 90 123.75 135
78.75 112.5 135 123.75 90
78.75 135 90 123.75 112.5
78.75 135 112.5 123.75 90
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90 75 120 135 120
90 135 120 75 120
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120 90 120 120 90
120 90 120 120 90
120 90 120 120 90
120 75 90 135 120
120 75 120 135 90
120 120 75 135 90
90 120 75 135 120
75 120 120 135 90
90 75 120 135 120

Here in the two degrees and in the part of three degrees, where the strings of two angles form 180 degrees, the question is over. The complexity of the parqueting modules here, come to the Type 2. But for the variants of № XV and those with the knots formed from three equal angles +1 different, the possibility of forming ne w types is possible. There are 36 numbers that need to be examined. Other possible tilings are probably here. The TABLE 1 is the algorithm for that.

Rice's research gives chance for discoveries connected to the creation of networks of hybrid pentagons, i.e. those belonging to more than one of the thirteen types and having specific networks of their own, apart from their corresponding ones. Thus, the general list of networks including the thirteen types, total to 58 (according to a letter from Rice to Schattschneider), which is to say the number of hybrid networks is 45. What's more, this only applies for networks from the first and second degree, plus networks III, V, and IX.

Indeed, Rice also started researching pentagons of the third degree, but she was not very optimistic and probably did not share the results with Schattschneider. Why? It's clear that to a large extent, the problem of hybrid forms remained a side-issue to her, and second, she had obviously ran into some contradictions with her own formula, which she was unable to overcome.

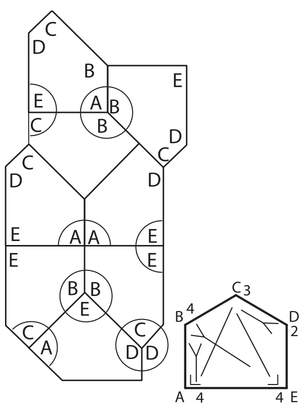


Figure 4

With the first degree pentagons there are no such contradictions. But at the second degree some double connections emerge that are somewhat confusing. This applies more importantly for the higher degrees.

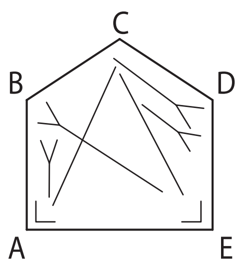


Figure 5

The network of eight-piece module shown on Figure 4 features seven different knots. Although it forms a network, at first sight it does not satisfy Rice's formula, i.e. the angles do not appear in the same numbers in all knots of the network. For a long time, finding such networks had been classified in the research, initially, as exceptions to Rice's formula. With their number growing, Rice's formula lost its sense. Incredibly, this seemed to enhance the area of possible networks.

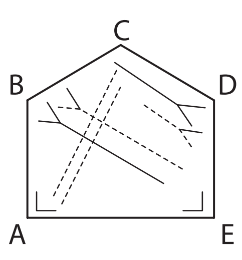


Figure 6

Yet everything returned to the proper framework with the discovery that by introducing double, triple, and so on links makes possible the restoration of angles numbers equality (Figure 5). In case this remains impossible, parquet forming is impossible too, (Figure 6), and here a new AC connection is needed.

Examining pentagons of higher degrees, even if not delivering a new parquet forming type, is absolutely necessary while searching for hybrid networks, which are as important for artists, designers, and architects, as the fifteen basic networks.

This study brought the following results:

First degree – 64 networks;
Second degree – 98 networks;
Third degree – 62 networks;
Fourth degree – 49 networks;
Fifth degree – 9 networks;
Sixth degree – 16 networks;
Eighth degree – 1 network;
Tenth degree – 1 network;
Total – 300 networks.

It's easy to notice that Rice's results not only lack networks of above the second degree, but also these of the first and second degree are, accordingly, 2 and 53, equals 55, while in this current research the number is 64 + 95 = 159, i.e. in this case the search delivers three times higher result. /I realized from my correspondence with Schattschneider that Marjory Rice had sent her other networks which are unfortunately not published so far./

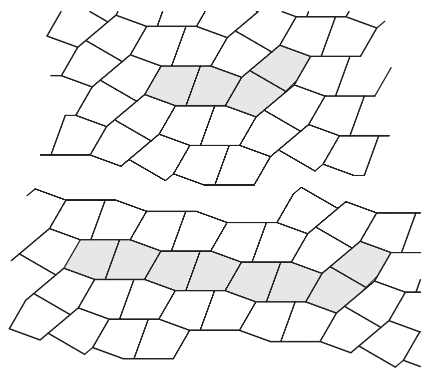


Figure 7

This might be explained by both obvious lapses in Rice's table, and her insufficient researches on parquet forms. The high results received in this current study are due to the profound analysis of parquet-forming creating opportunities, as well as to improve the technologies of combination.

In general, all networks are unchangeable or changeable.

In the case of changeable networks, some of the elements of the parquet-forming module may be repeated, tripled, etc., as the network acquires a new graphic expression without modifications to the parquet-forming pentagon and its knots. Now it arrives at a new network with the same data and of the same degree (Figure 7).

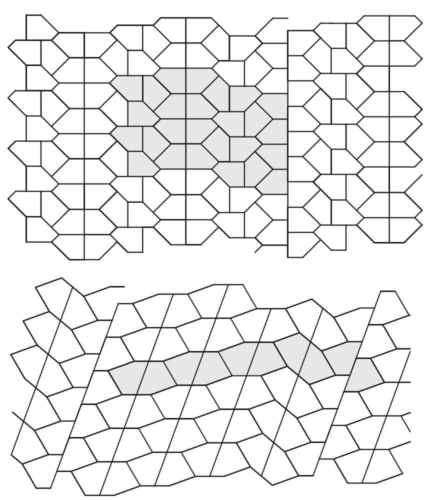


Figure 8

In other networks, typical with unbroken parallel lines in their compositional structure, sliding of the belts between these lines is possible, and even in different scales within one and the same network. These are again other networks, but with the same parameters (Figure 8).

In the third case, the inversion combination of the module forms a new module with

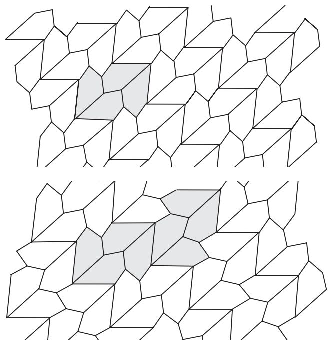


Figure 9

doubled number of the pieces, but with the same parameters. Such is the case of networks VII,VIII,X,XI,XII,XIII and VI the biomorph characteristics of which Rice overlooked. (Figure 9).

Fourth case – we should point out that in network I, two distinct forms of its deserving differentiation, have not been counted: network "O", where the two parallel sides of the pentagon are equal, and network I^2, where the pentagon has two pairs of parallel sides. These two types, and especially the latter, are distinctive for their architectonic features. The I^2 network also gives a combination of two hexagons, and the "O" type – sliding networks (Figure 10).

Fifth case. Totally unexplored remain the three types of rosettes of first and third degree – accordingly, of diagonal and biaxial equilateral hexagons, and parquet-forming mirror binomials, as well as their variations: solid, with central opening, and correspondingly, open or closed. The first "rosette" of Rice is actually just a diendric axial construction. Schattschneider mentions a group of Australian students who,

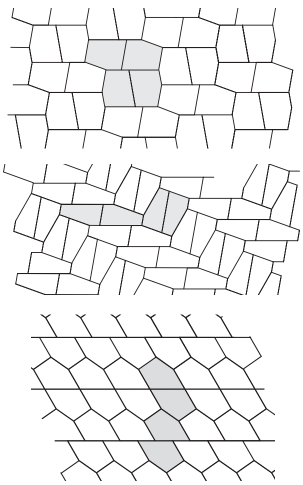


Figure 10

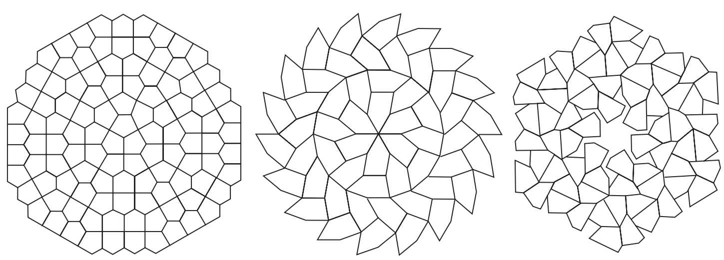


Figure 11

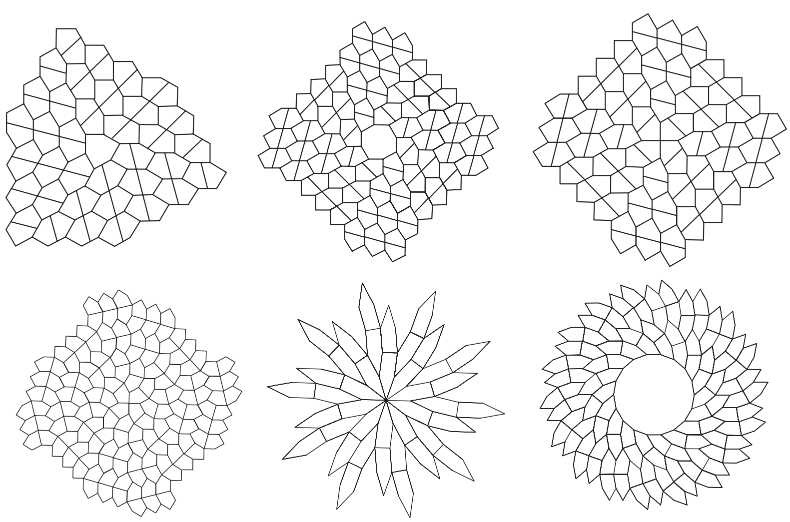


Figure 12

under Hirshhorn's lead, discovered two rosettes of equilateral unpaired pentagons and the pentagon of Hash (Figure 11), but these were not infinite. (Later on it was proved that Hirshhorn's rosette might be continued to infiniteness in four variants.) To these we may add the rosette of Zuka, and the wonderful polycentric rosette of Rice, but these are rosettes of unique pentagons. Which is to say, that the infinite model rosettes, which are extremely important for mastering centric oriented spaces, remained unknown (Figure 12).

In fact, our entering the higher degrees of networks was a result not of preliminarily composed tables, but of researching changeable networks. The major part of them when changed, pass into another degree, receiving new knots. Of particular interest are those networks which change in two directions and pass through different networks (Figure 13).

Thus, through modification of a new network we determine the parquet-forming group and hence discover, by using Rice's "hieroglyphs" we can outline all members of the subgroup. Another method for discovering high-degree parquet-forming modules is through upgrading low-degree parquet-forming or non-forming ones (Figure 14). As already stated, networks III, V, and IX are also high-degree ones – III and IX – of third degree, and V – of sixth. (Marjory Rice had irregularly placed all the three in third degree.) Besides, network III (which, in fact, is an detailed drafted network of regular hexagons) is not only bimorph (formed through simple translation or mirror binomial translation (Figure 15)), but at the translation of inversion or diagonal binomi-

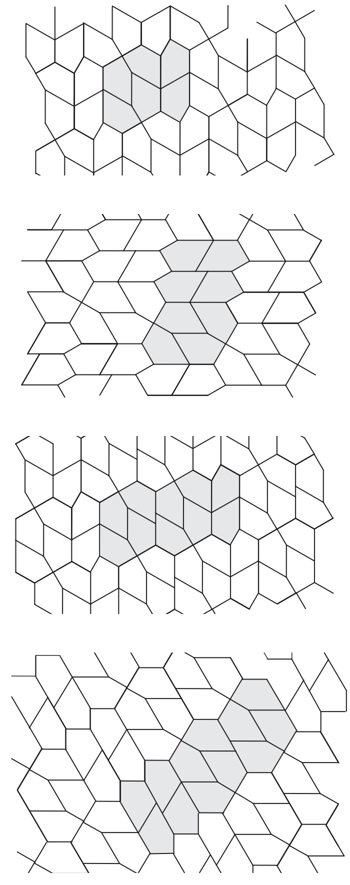


Figure 13

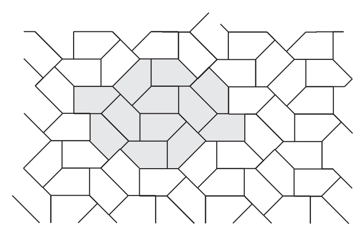


Figure 14

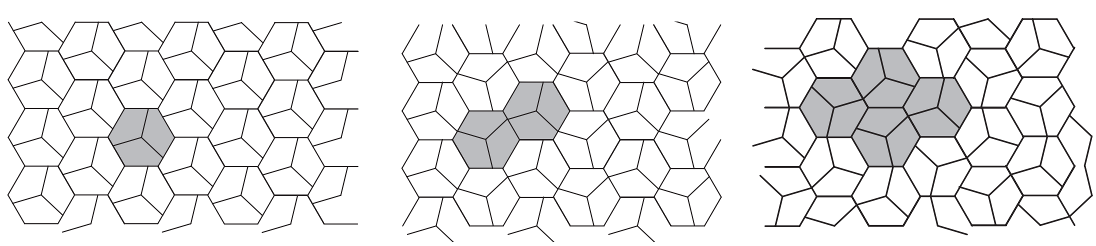


Figure 15

als, it gives a new network type. So, the bimorph network III and its two related polymorph networks provide endless possibilities for form-building in the range from the third to sixth (and to 11) degree.

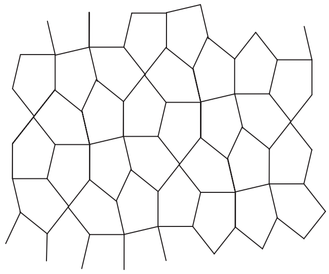


Figure 16

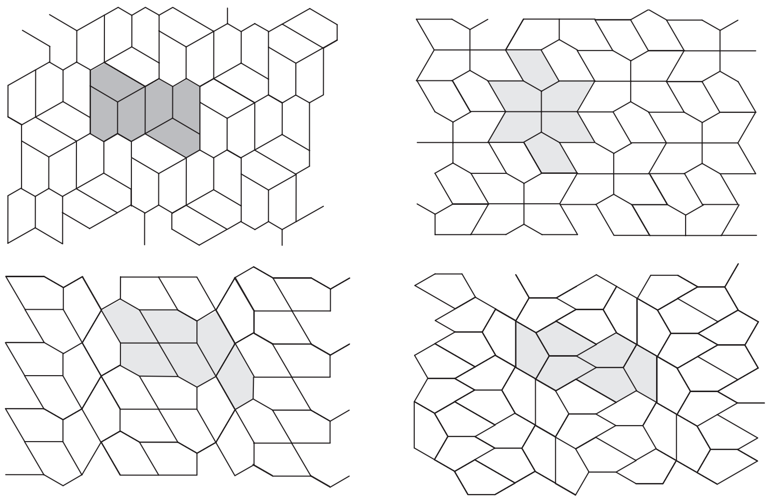


Figure 17

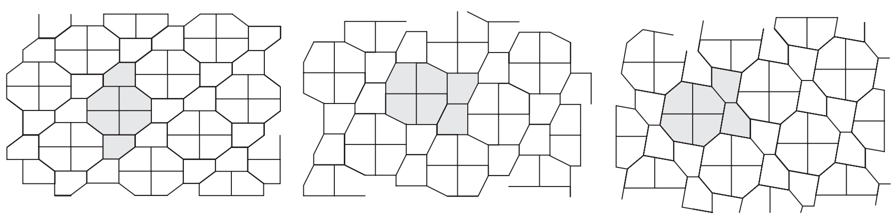


Figure 18

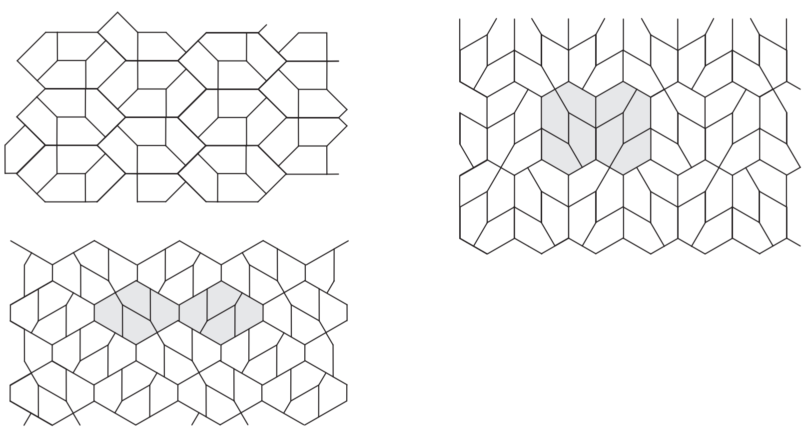


Figure 19

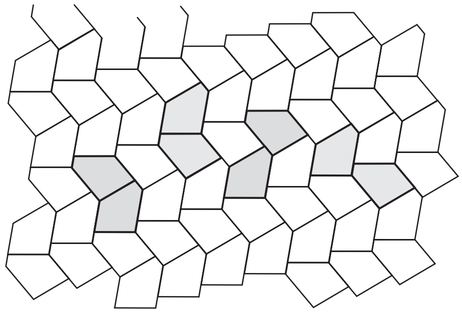


Figure 20

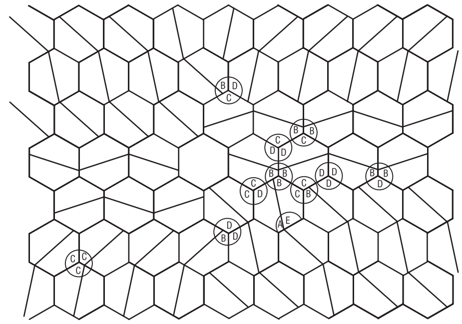
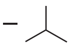


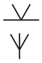
Figure 21

Special attention is paid to the networks featuring a knot of three equal angles (of 120º), whose signature we add to Rice's hieroglyphs –  (Figure 17).

We also add the following signatures:

 – 6 equal angles of 60º;

 – 3 equal angles of 60º;

 – 3 equal angles + 1 = 360º.

Also it should be pointed out that James' network, which is in fact unchangeable, and has two graphic analogues (Figure 18).

Another occurrence that draws attention is the so-called "artistic games", allowing greater freedom for creative searching. These are not only possible in network III and the sliding networks, but also in the networks forming biaxial hexagons (Figure 19), as well as in more complicated symmetrical configurations. Such "games" are, of course, possible in each and any network. Given a parquet-forming module (even in the case of parquet-forming binomials) (Figure 20) is never unequivocal, which module type shall be accented and determined basic, depends on the concrete composition demands only. It could even reach to illusionary presentation of monoendric networks as polyendric.

All these networks allow transformations while retaining their monoendricity (Escher), or deformations turning them into a plastic material in the hands of an artist (Vasarely).

And another peculiarity: low-degree networks are usually networks of types, i.e. pentagons in any corresponding parquet-forming module might be endlessly modified according to the requirements of their subgroups. These are relatively simple constructions with countless possibilities for changes. Besides, they may be subject to the above mentioned modifications within or outside the sub-type frames. The high-degree networks, which, as possible combinations of 3, 4, 5, 6, etc. knots are far more than the low-degree ones, and in fact decrease in numbers with the growing of the degree and become unique and unchangeable. They are complex, but fixed constructions.

But if the networks' degree is mathematically infinite, then which is the actual last degree? A profound analysis reached to the network of binomials in regular hexagons, which in fact exhausts its possible variations in the ninth degree with eleven knots (Figure 21).

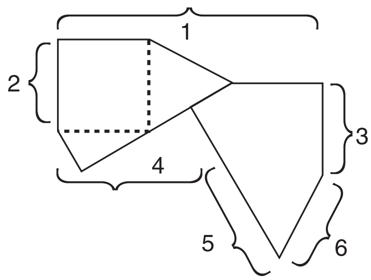


Figure 22

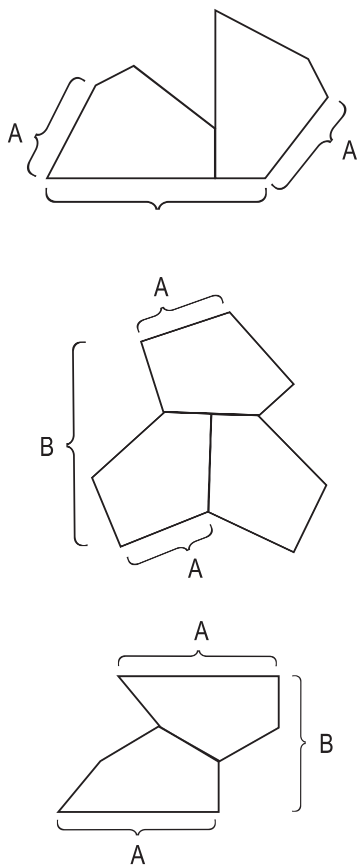


Figure 23

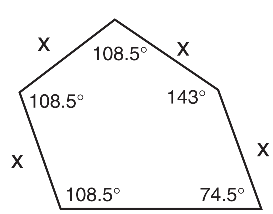


Figure 24

The methodical and profound research on all possible groups that was completed for the second and partially for the third degree is not necessary for the higher degrees, as they are mostly developments or combinations of modules from the lower degrees. In this regard this research is complete and comprehensive and covers this theme (with some possible lapses allowed). Fortunately, in the hands of an artist it is inexhaustible, and not only regarding ceramic floorings or the directions of Escher and Vasarely, but most of all in the opportunities of each network for "artistic games".

Yet the artistic embrace of the mathematical problem would not be impossible without the contributions of Marjory Rice (the rule on networks, the graphic information, and numeration).

The improvement of the technique of combining, though, comes from knowing the laws of symmetry (mirror, diagonal, inverse, and translational), and the criterion of Conway on the parquet forming ability:

"A figure gets realized as a prototype for monoendric network if it's outlines might get decomposed into six parts, two of which are equal and parallel, and four – in diagonal symmetry." (Straight lines are included here.)

Although it was known that the pentagon of Hash parquets limitedly and not infinitely, one of its combinations was found to fully meet the above criterion, but is still not forming parquet (Figure 22). At this exception the two equal and parallel sides are adjacent to one diagonal side. Hence, at decomposing into six parts this should be avoided and this is the new requirement. And what happens when the figure is decomposed into three, four or five parts? With three parts (derivative of a scalene triangle) neither equality nor parallelism of sides is required (as the latter is impossible), but only their diagonal symmetry. With four parts (scalene quadrangle) the requirements are the same (with possible parallelism of the sides). Conway's criterion is only precise for decomposing into five parts. Thus cleared, this criterion allows quick and faultless determination of the parquet forming shapes. Similar binomial uhich is a not parqueting modul is at Figure 23.

It is natural for all artists (and especially for designers) to emphasise the importance of those pentagons characteristic for their high polymorphism. For this purpose, research and tables have been made, revealing the parquet-forming capacities of all the 300 networks, with the undisputed top being the pentagon (Figure 24) delivering over 50 networks.

And finally, we will make a summary: parquet-forming irregular pentagons cover infinite planes in two manners, as follow:

A – through development along two rectilinear directions;

B – through rotation of the parquet forming module around one centre.

For the first case, among the many wonderful networks we shall point out to the sliding of type "O", and for the second – the rosettes of first and third degree.

Both cases take us directly on a concrete architectural environment – pavements and panorama alleys (Sidewalks "1", "2", and "3"), as well as in centric-oriented interior and exterior spaces (Rosettes A, Rosettes B, Tableau "1").

All the other networks, and especially these of I^2, may be integrated or employed with endless variety in the free urban and park spaces.

Besides, to the richness of the practical capacities of irregular convex pentagons we will add the already mentioned "artistic games", where the activated visualisation of the different types of parquet-forming modules in one and the same network, or parts of them, or of more complex formations, brings forward an endless and surprising diversity.

In the hands of an artist, through the insertion of different colour, texture, and material, pentagons, geometrical networks turn into subjects of genuine compositional adventure.

Such richness could be an adequate contemporary transcription of such a classical masterpiece as Il Mare – the floor mosaic of the San Marco Basilica in Venice.

We do hope that the monoendric infinite networks of irregular pentagons will see a wide application and enhance the limited capacities of rectangular plates.

Other research and studies by the author have been published in:

Problems of the Arts magazine, issue 1, 1998;

AIMC Catalogue – 7th Congress, 2000, Ravenna, Italy;

Solo Mosaico magazine, 2008, Saint Petersburg, Russia;

ORNAMENT, 2014, Arhimed, Sofia, Bulgaria;

MY WAY, 2015, Arhimed, Sofia, Bulgaria.

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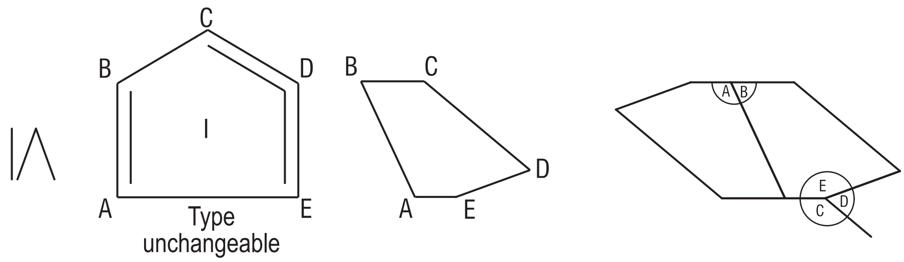

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